Chapter 01: The Real and Complex Number Systems

一、有理数的不完备/稀疏性

Ex 0: Pf: Equation $p^2=2$ is not satisfied by any rational number p.

Pf: If $\exists\, p\in\mathbb{Q}$, s.t. $p^2=2$, let $p=\frac{m}{n}$, where $n\neq 0$, $(m,n)=1$

Thus we have: $m^2=2n^2 \Rightarrow m^2$ is even $\Rightarrow m$ is even $\Rightarrow 2|m$

Let $m=2k$, $k\in\mathbb{Z}$

$\therefore 2n^2=4k^2$

$n^2=2k^2 \Rightarrow n$ is even $\Rightarrow 2|n$

$2|m \Rightarrow 4|m^2 \Rightarrow 2|n^2 \Rightarrow$ (Alternative Way)

$\therefore 2|m,\ 2|n \Rightarrow (m,n)=2$, Contradict to our assumption $(m,n)=1$

$\therefore \nexists\, p\in\mathbb{Q}$, s.t. $p^2=2$

QED.

更进一步的证明有理数并不能完美覆盖 $\mathbb{R}$ (有空隙)

有序集 (Ordered Set)

1.5 Definition Let S be a set. An *order* on S is a relation, denoted by $<$, with the following two properties:

(i) If $x\in S$ and $y\in S$ then one and only one of the statements → 三歧性

$$x<y,\qquad x=y,\qquad y<x$$

is true.

(ii) If $x, y, z\in S$, if $x<y$ and $y<z$, then $x<z$. → 传递性

集合的序

1.6 Definition An *ordered set* is a set S in which an order is defined.

For example, Q is an ordered set if $r<s$ is defined to mean that $s-r$ is a positive rational number.

集合的分划

对于给定的全序集 A，及其中元素 $x\in A$，我们可通过 x 将 A 分划为两个非空集合 P & P'，其中 $P\cup P'=A$ 且 P 中元素均在 x 之前，P' 中元素均在 x 之后。通常称 P 为分划的下组，P' 为上组，记为 $P|P'$。

令 $A=\mathbb{Q}$，合法情况为：

1° P 中有最大数，P' 中无最小数 $\Rightarrow P=(-\infty,x]\quad P'=(x,+\infty)$

2° P 中无最大数，P' 中有最小数 $\Rightarrow P=(-\infty,x)\quad P'=[x,+\infty)$

不合法情况为：

3° P 中有最大数，P' 中有最小数

Ex1: Pf：设 $\max P=a$，$\min P'=b$，则由定义，$a,b\in\mathbb{Q}$

$$\Downarrow$$

$$a<\frac{a+b}{2}<b\quad\in\mathbb{Q}$$

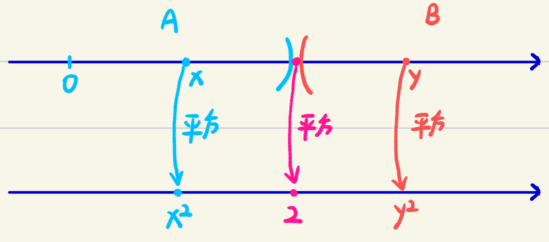
$$\text{而 }\frac{a+b}{2}\notin P\ \wedge\ \frac{a+b}{2}\notin P'\text{，与 }P\cup P'=\mathbb{Q}\text{ 矛盾}$$

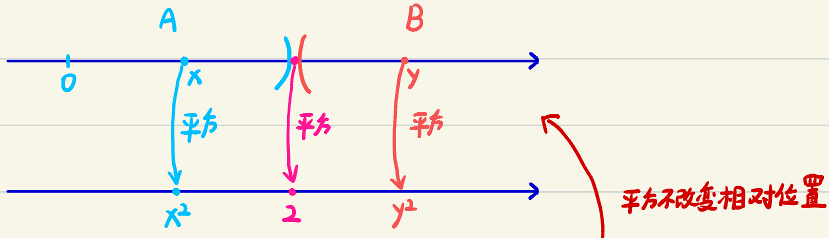
(除非考虑 $P=(-\infty,x]\quad Q=[x,+\infty)$) QED.

4° P 中无最大数，P' 中无最小数 (Dedekind Cut)

$$\text{令}\begin{cases}A=\{x\in\mathbb{Q}\mid x^2<2\text{ or }x<0\} & \Rightarrow \text{All }\mathbb{Q}\text{ in }(-\infty,\sqrt{2})\\ B=\{y\in\mathbb{Q}\mid y^2\geq 2\text{ and }y\geq 0\} & \Rightarrow \text{All }\mathbb{Q}\text{ in }[\sqrt{2},+\infty)\end{cases}\quad\text{简化：}\begin{cases}A=\{p\in\mathbb{Q}^+\mid p^2<2\}\\ B=\{p\in\mathbb{Q}^+\mid p^2>2\}\end{cases}$$

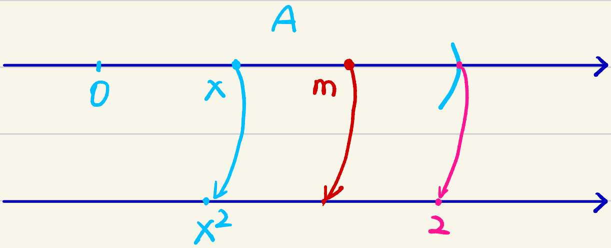
Ex2: Pf：A 无最大数，B 无最小数，即由 $x^2=2$ 确定的 $x\notin\mathbb{Q}$ (在 $x^2=2$ 的 x 处，$\mathbb{Q}$ 有一个"洞")





Lemma: If $y>x>0 \iff y^2-x^2=(y+x)(y-x)>0 \iff y^2>x^2$

欲证 A中无 max $\iff \forall x\in A, \exists m\in A, s.t. m>x$



借助增量 r，即找 $r>0$, s.t. $m=x+r$ 满足 $m\in A \Rightarrow (x+r)^2<2$

$\iff \underbrace{r^2}+2xr+x^2<2 \quad (*)$

↳ 仍化平方项, 当 $0<r<1$ 时,

若 $r+2xr+x^2<2$ (更强命题成立), 则 $(*)$成立, 即

$$r<\frac{2-x^2}{2x+1}$$

取 $r=\frac{2-x^2}{m(2x+1)}<1 \ (m>0)$, 即 $2-x^2<2mx+m \iff \underbrace{x^2+2mx}_{} \overset{\wedge}{\ } > \underbrace{2-m}_{}$ 0即可 不妨取 $m=2$, 即

$$r=\frac{2-x^2}{2(2x+1)}$$

即对 $\forall x\in A$, 均$\exists\ m=x+\frac{2-x^2}{2(2x+1)}\in A$, 且 $m>x$.

QED.

针对集合B的命题同理可证

另一种思路：

$A=\{p\in\mathbb{Q}^+ \mid p^2<2\}$ 无 max / $B=\{p\in\mathbb{Q}^+ \mid p^2>2\}$ 无 min

$\Leftrightarrow$ 构造 $q\in A$，满足 $q>p$ / $q\in B$，满足 $q<p$

条件① 条件②

条件①的构造：$q^2<2$ / $q^2>2$

$\Leftrightarrow q^2-2<0$ / $q^2-2>0$

q^2 与 2 的关系与 p^2 与 2 的关系同步

即可符合条件①

令 $q^2=2+\dfrac{p^2-2}{\square}$ → 恒正

□ 恒正，不妨令 $\square=\Delta^2$，即

$$q^2=2+\frac{p^2-2}{\Delta^2}=\boxed{\frac{p^2+2\Delta^2-2}{\Delta^2}}$$

最好是完全平方式 — 条件①

$$q^2-p^2=(2-p^2)\left(1-\frac{1}{\Delta^2}\right)$$

A	+	+
B	−	−

恒正 → $\Delta>1$ — 条件②

条件①：$q^2=\dfrac{p^2+2\Delta^2-2}{\Delta^2}$ 为完全平方

条件②：$\Delta>1$

条件① $\Leftrightarrow$ $p^2+2\Delta^2-2$ 为完全平方

不妨设 $\Delta=mp+n$

则 $p^2+2\Delta^2-2=(2m^2+1)p^2+4mnp+(2n^2-2)$ 为完全平方.

即 $\begin{cases}2m^2+1\\2n^2-2\end{cases}$ 完全平方 & $\left(\dfrac{4mn}{2}\right)^2=4m^2n^2=(2m^2+1)(2n^2-2)$

$=4m^2n^2-4m^2+2n^2-2$

$\Leftrightarrow n^2=2m^2+1\quad(*)$

只要 $(*)$ 成立，✓，$2n^2-2=4m^2$ 亦为完全平方

$\Downarrow$

只需找到 $n^2=2m^2+1$ 的整数解，即可符合条件①

显见，$\begin{cases}m=2\\n=3\end{cases}$ 为 $n^2=2m^2+1$ 的一组解，即 $\Delta=2p+3>1$（同时符合条件②）

Pell 方程：$x^2-ny^2=1$ $\begin{cases}n\text{为完全平方数——仅有平凡解}(\pm1,0)\\n\text{为非完全平方数——总有非平凡解}\end{cases}$

$\sqrt{2}=1+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cdots}}}$　$\sqrt{2}$ 的渐近连分数形式为 $\left[\frac{1}{1};\frac{3}{2},\frac{7}{5},\frac{17}{12}\cdots\right]$

$\begin{cases}x=3\\y=2\end{cases}$ 为该 Pell 方程的最小解，则通解有：$\begin{cases}x_{i+1}=x_1x_i+ny_1y_i\\y_{i+1}=x_1y_i+y_1x_i\end{cases}$ 或 $x_i+y_i\sqrt{n}=(x_1+y_1\sqrt{n})^i$

$$x_n=\frac{(3+2\sqrt{2})^n+(3-2\sqrt{2})^n}{2}\qquad y_n=\frac{(3+2\sqrt{2})^n-(3-2\sqrt{2})^n}{2\sqrt{2}}$$

$n^2=2m^2+1$

$n=3\quad 17\quad 99\quad 577\quad 3363\ \cdots\cdots$

$m=2\quad 12\quad 70\quad 408\quad 2378\ \cdots\cdots$

$$q^2=\frac{p^2+2\Delta^2-2}{\Delta^2}\quad \&\quad \Delta=2p+3$$

$$q^2=\frac{p^2-2+2\times(2p+3)^2}{(2p+3)^2}=\frac{9p^2+24p+16}{(2p+3)^2}=\frac{(3p+4)^2}{(2p+3)^2}$$

$\therefore$ 取 $q=\dfrac{3p+4}{2p+3}$，即有 $\begin{cases} p^2<2, & p^2<q^2<2 \\ p^2>2, & 2<q^2<p^2 \end{cases}$

Q.E.D.

验证

$$q^2-2=\frac{(3p+4)^2-2(2p+3)^2}{(2p+3)^2}=\frac{p^2-2}{(2p+3)^2}\Rightarrow q与p同属一集合$$

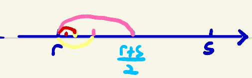
$$q-p=\frac{3p+4-2p^2-3p}{2p+3}=\frac{2(2-p^2)}{2p+3}\Rightarrow\begin{cases} p^2<2, & q>p \\ p^2>2, & q<p \end{cases}$$

符合要求

$\Rightarrow$ 即便有理数可无限二分，即 $\forall\, r<s\in\mathbb{Q},\ \frac{r+s}{2}\in\mathbb{Q}$

$\Downarrow$

但 $\mathbb{Q}$ 系统并未填满 $\mathbb{R}$，仍有空隙，并不完备



二、集合的界

1.7 Definition Suppose S is an ordered set, and $E \subset S$. If there exists a $\beta \in S$ such that $x \leq \beta$ for every $x \in E$, we say that E is *bounded above*, and call β an *upper bound* of E.

Lower bounds are defined in the same way (with $\geq$ in place of $\leq$).

S为一有序集合，E为S上一子集。若∃S中元素$\beta \in S$，对$\forall x \in E$，均有$x \leq \beta$，则称β为E的上界；若对$\forall x \in E$，均有$x \geq \beta$，则称β为E的下界。

（S一般为我们的研究范围，比如R/Q ……）

N.B. β可能$\in E$，也可能$\notin E$

1.8 Definition Suppose S is an ordered set, $E \subset S$, and E is bounded above. Suppose there exists an $\alpha \in S$ with the following properties:

(i) α is an upper bound of E. → α自身要是集合的上(下)界

(ii) If $\gamma < \alpha$ then γ is not an upper bound of E. → 比α再小(大)的，就不是界

Then α is called the *least upper bound of E* [that there is at most one such α is clear from (ii)] or the *supremum of E*, and we write

$$\alpha = \sup E.$$ —最小上界 —上确界

The *greatest lower bound*, or *infimum*, of a set E which is bounded below is defined in the same manner: The statement

$$\alpha = \inf E$$ —最大下界 —下确界

means that α is a lower bound of E and that no β with $\beta > \alpha$ is a lower bound of E.

e.g. $\begin{cases} A=\{p \in Q^+ \mid p^2<2\} \\ B=\{p \in Q^+ \mid p^2>2\} \end{cases}$

$A=\{p \in Q^+ \mid p^2<2\}$ — A的上界恰为B — B无min — A无sup

$B=\{p \in Q^+ \mid p^2>2\}$ — B的下界恰为A — A无max — B无inf

上确界性 & 下确界性

1.10 Definition An ordered set S is said to have the *least-upper-bound property* if the following is true:

If $E \subset S$, E is not empty, and E is bounded above, then $\sup E$ exists in S.

上确界性：在序流S下，若 $\begin{cases} Z \neq \emptyset \\ Z\text{有上界} \end{cases}$，则 Z有上确界 （有上界必有上确界）

⇑ 只是完备的（没洞）

1.11 Theorem *Suppose S is an ordered set with the least-upper-bound property, $B \subset S$, B is not empty, and B is bounded below. Let L be the set of all lower bounds of B. Then*

$$\alpha = \sup L$$

exists in S, and $\alpha = \inf B$.

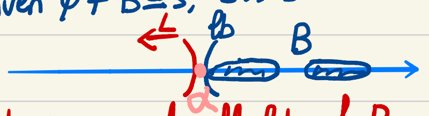
In particular, $\inf B$ *exists in S.*

引出的上确界性与下确界性对偶

Ex3: Pf: lub性与glb性对偶

即已知 S has l.u.b property (ordered set — 可排在轴上)

Given $\emptyset \neq B \subseteq S$, B is b.d.d below



考虑 L = set of all l.b. of B

(i) $L \neq \emptyset \Leftarrow$ B is b.d.d. below

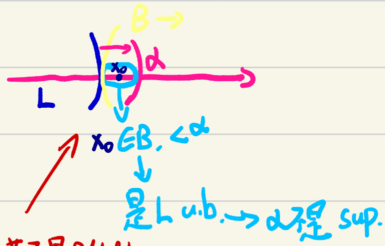
(ii) L is b.d.d. above — $\forall$ B中元素为L上界 ($B \neq \emptyset$)

⟹ sup L 存在，设为 $\alpha \in S$

猜想：$\sup L = \alpha = \inf B$

L是B下界构成的集合，$\alpha = \sup L$，Pf: $\alpha = \inf B$

L → $\begin{cases} S_1: \alpha \text{是B的l.b.} \\ S_2: \alpha \text{是B的最大l.b.} \begin{cases} 1° \text{若} l \in L, \text{则} l \leq \alpha \\ 2° \forall \gamma > \alpha, \gamma \notin L \,/\, \gamma\text{不是B的l.b.} \end{cases} \end{cases}$



S_1: $\alpha = \sup L \iff \forall x \in B, \alpha \leq x$ (α是B的l.b.) 若不是B的l.b.

Pf: 若$\exists x_0 \in B, x_0 < \alpha$，因$\forall x_0 \in B$，$x_0$均为L的u.b.，且$x_0 < \alpha$，与$\alpha = \sup L$矛盾.

Alt. Pf: $\alpha = \sup L$，若$\exists x_0 \in B, x_0 < \alpha$，则$x_0$不是L的上界. 而$x_0 \in B$，B为L的上界集，矛盾.

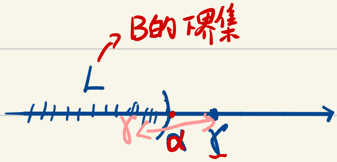
S_2: 若$\exists \gamma > \alpha$，且γ为B的l.b. $\Rightarrow \gamma \in L$

L → $\alpha = \sup L \iff \forall \delta \in L, \alpha \geq \delta$

$\gamma > \alpha$

→ $\gamma > \alpha \geq \forall \delta \in L$

$\Rightarrow \gamma \notin L$ 矛盾.



S_1、S_2均满足，因此lub与glb对偶

QED.

L是B下界构成的集合，$\alpha=\sup L$，Pf：$\alpha=\inf B$

L是B的下界集

$L\neq\phi$

L有上界（$\forall x\in B$为L的u.b.）

S具有lub性

$\sup L$存在，设$\alpha=\sup L$

$\Downarrow$

$\forall \delta\in L,\ \delta\leq\alpha$

设$\gamma=\inf B$

则$\gamma\in L$（γ为B下界）

$\therefore\ \gamma\leq\alpha$

下证$\gamma<\alpha$不成立即可

因$\gamma=\inf B$，我们有

$\forall \delta\in L,\ \delta\leq\gamma$（$\gamma$为L上界）

若$\gamma<\alpha$，则与$\alpha=\sup L$矛盾

因此$\gamma=\alpha$，即$\sup L=\inf B$

QED

三、数域

1.12 Definition A *field* is a set F with two operations, called *addition* and *multiplication*, which satisfy the following so-called "field axioms" (A), (M), and (D):

(A) Axioms for addition

(A1) If $x \in F$ and $y \in F$, then their sum $x + y$ is in F.
(A2) Addition is commutative: $x + y = y + x$ for all $x, y \in F$.
(A3) Addition is associative: $(x + y) + z = x + (y + z)$ for all $x, y, z \in F$.
(A4) F contains an element 0 such that $0 + x = x$ for every $x \in F$.
(A5) To every $x \in F$ corresponds an element $-x \in F$ such that

$$x + (-x) = 0.$$

(M) Axioms for multiplication

(M1) If $x \in F$ and $y \in F$, then their product xy is in F.
(M2) Multiplication is commutative: $xy = yx$ for all $x, y \in F$.
(M3) Multiplication is associative: $(xy)z = x(yz)$ for all $x, y, z \in F$.
(M4) F contains an element $1 \neq 0$ such that $1x = x$ for every $x \in F$.
(M5) If $x \in F$ and $x \neq 0$ then there exists an element $1/x \in F$ such that

$$x \cdot (1/x) = 1.$$

(D) The distributive law

$$x(y + z) = xy + xz$$

holds for all $x, y, z \in F$.

Field: $+$, $\times$

1. 封闭性
2. 交换律
3. 结合律
4. 单位元
5. 逆元

"+" & "×" → 分配律

⚠通过Axiom证明性质时,要多用代换—$0=x+(-x)$ / $x=0+x$ / $1=x\cdot(\frac{1}{x})$ 搭配其他题目前提等式

有序域

1.17 Definition An *ordered field* is a *field* F which is also an *ordered set*, such that

(i) $x + y < x + z$ if $x, y, z \in F$ and $y < z$,
(ii) $xy > 0$ if $x \in F$, $y \in F$, $x > 0$, and $y > 0$.

If $x > 0$, we call x *positive*; if $x < 0$, x is *negative*.

有序域 { ①是有序集 ②同加不变序 ③同正乘还正

→ 不等式的常见法则对任一有序域均成立

实数域

We now state the *existence theorem* which is the core of this chapter.

1.19 Theorem *There exists an ordered field R which has the least-upper-bound property.*

Moreover, R contains Q as a subfield. (R上的"+"/"×"与Q上的一致)

The second statement means that $Q \subset R$ and that the operations of addition and multiplication in R, when applied to members of Q, coincide with the usual operations on rational numbers; also, the positive rational numbers are positive elements of R.

The members of R are called *real numbers*.

实数的阿基米德性(Archimedean Property)

(a) *If $x \in R$, $y \in R$, and $x > 0$, then there is a positive integer n such that*

$$nx > y.$$

任一正实数不断累加自身后，都能比另一个给定实数大

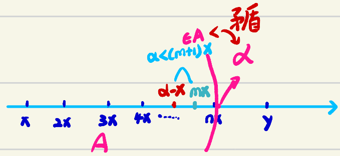
→ 令 $x=1$，则证明了 Z^+ 非 bounded

Ex4: Pf: $\forall x\in R^+, \exists n\in Z^+$, s.t. $nx>y$ for $\forall y\in R$

反证：若对 $\forall n\in Z^+$，均有 $nx\le y$，则 y 为 $A=\{nx \mid n\in Z^+\}$ 的 u.b.

因 $nx\in R$，因此具有 lub 性，即 $\sup A$ 存在，不妨设为 α

即对 $\forall n\in Z^+$，$nx\le\alpha\le y$



$\because x>0$

$\therefore \alpha - x < \alpha \quad (\alpha=\sup A)$

$\therefore \alpha - x$ 非 A 的 u.b.，即 $\exists m\in Z^+$，s.t. $mx>\alpha-x$

$\Leftrightarrow \alpha<(m+1)x$，而 $m+1\in Z^+$，即 $(m+1)x\in A$，与 α 为 A u.b. 矛盾

$\therefore \exists n\in Z^+$, s.t. $nx>y$ for $x\in R^+$, $y\in R$

有理数在R上的稠密性 (比R自身的稠密性更强)

Ex5: Pf: (b) *If $x\in R$, $y\in R$, and $x<y$, then there exists a $p\in Q$ such that $x<p<y$.*

证明1：基于实数的 Decimal Expansion ($\forall x\in R$ 都是不断延伸列(随 $k\nearrow$ 而延申) $n_0+\frac{n_1}{10^1}+\cdots+\frac{n_k}{10^k}$ $(k=0,1,\cdots)$ 的 sup)

1.22 Decimals We conclude this section by pointing out the relation between real numbers and decimals.

Let $x>0$ be real. Let n_0 be the largest integer such that $n_0\le x$. (Note that the existence of n_0 depends on the archimedean property of R.) Having chosen $n_0, n_1, \ldots, n_{k-1}$, let n_k be the largest integer such that

$$n_0+\frac{n_1}{10}+\cdots+\frac{n_k}{10^k}\le x.$$

Let E be the set of these numbers

(5) $$n_0+\frac{n_1}{10}+\cdots+\frac{n_k}{10^k}\qquad (k=0,1,2,\ldots).$$

Then $x=\sup E$. The decimal expansion of x is

(6) $$n_0\cdot n_1n_2n_3\cdots.$$

Conversely, for any infinite decimal (6) the set E of numbers (5) is bounded above, and (6) is the decimal expansion of $\sup E$.

令 $x=\boxed{a_0.a_1a_2\cdots a_i}a_j\cdots$

$y=\boxed{a_0.a_1a_2\cdots a_i}a_n\cdots$

Simply take

$$p=\frac{a_0.a_1a_2\cdots a_ia_j+a_0.a_1a_2\cdots a_ia_m}{2}$$

QED

(b) *If $x \in R, y \in R$, and $x < y$, then there exists a $p \in Q$ such that $x < p < y$.*

证明$_2$：基于 Well-ordering Principle —— $\forall Z^+$的非空子集必有最小元

已知 $y>x$

1° $y>0>x$，显然取 $p=0$ 即可

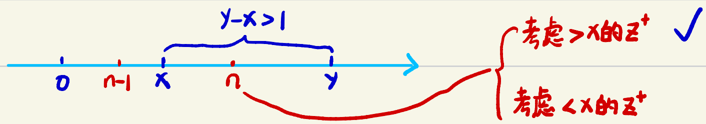
2° 不失一般性，设 $y>x>0$ （$x<y<0$ 为对偶命题）

(i) 若 $y-x>1$，则 $\exists\, n\in N$, s.t. $x<y<1$，取 $x<n=[x]+1<y$ 即可

(ii) 若 $y-x<1$，依阿基米德性，$\exists\, m\in Z^+$, s.t. $m(y-x)>1$，则 $\underbrace{\exists\, n\in N, \text{s.t. } mx<n<my}_{\text{化归到}3\ 1^\circ}$

$\hookrightarrow x<\frac{n}{m}<y$，即 $\exists\, p=\frac{n}{m}$

(i) 的非构造性证明：



不妨令 $A=\{p\in N \mid p>x\}$，由上述讨论（Z^+ unbounded）可知 $A\neq\phi$

因此，由 Well-ordering Principle，A 中存在最小元，不妨设为 n，即

$n-1\notin A \iff n-1\leq x \iff n\leq x+1$

而 $y-x>1 \iff \underline{y>x+1\geq n>x}$

$\hookrightarrow y>n>x$，n 即为所求

QED

(b) *If $x \in R$, $y \in R$, and $x < y$, then there exists a $p \in Q$ such that $x < p < y$.*

证明$_3$: Rudin的证明:

(b) Since $x < y$, we have $y - x > 0$, and (a) furnishes a positive integer n such that

$\mathbb{Z}^+, -\mathbb{Z}^+$ unbounded

$$n(y - x) > 1.$$ 扩增 $y-x$ 间距

Apply (a) again, to obtain positive integers m_1 and m_2 such that $m_1 > nx$, $m_2 > -nx$. Then

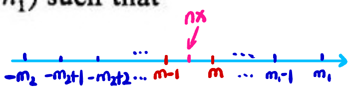
m_1: $\exists \mathbb{Z} > nx$

m_2: 并非 $\forall \mathbb{Z} > nx$

$$-m_2 < nx < m_1.$$ 约束 nx 范围 (Rudin并未讨论 nx 符号)

Hence there is an integer m (with $-m_2 \le m \le m_1$) such that

$$m - 1 \le nx < m.$$



If we combine these inequalities, we obtain

若$\exists$此m, 则 $nx < m$,

$$nx < m \le 1 + nx < ny.$$

$m-1 \le nx$

$\Leftrightarrow m \le nx+1 < ny$

$nx < m < ny$

$\Leftrightarrow x < \frac{m}{n} < y$ 证毕

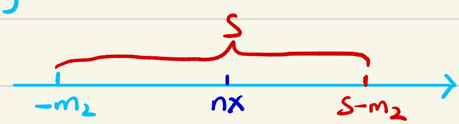
Since $n > 0$, it follows that

$$x < \frac{m}{n} < y.$$

This proves (b), with $p = m/n$.

Pf: m存在 (Trivial: 令 $m = [nx] + 1$)

令 $S = \{s \in N \mid s - m_2 > nx\}$



显然 $S \ne \phi$, 因 $m_1 + m_2 \in S$

故由Well-ordering Principle, S存在最小元, 不妨设为 S_0, 则有

$$\begin{cases} S_0 - m_2 > nx \\ S_0 - 1 - m_2 \le nx \end{cases} \Rightarrow \text{令 } m = S_0 - m_2, \text{则 } m-1 \le nx < m, \text{证毕}$$

$\mathbb{R}$中根的存在性

1.21 Theorem *For every real $x > 0$ and every integer $n > 0$ there is one and only one positive real y such that $y^n = x$.*

This number y is written $\sqrt[n]{x}$ or $x^{1/n}$.

Ex6: Pf Theorem 1.21

S_1: 解的唯一性

若 $y^n = x$ 有超过一个的解，不失一般性，记其中两个为 y_1 和 y_2，则 $y_1^n = y_2^n = x$

若 $y_1 \neq y_2$，则 $y_1 > y_2 > 0$ 和 $y_2 > y_1 > 0$ 必有一个成立，即 $y_1^n > y_2^n > 0$ 和 $y_2^n > y_1^n > 0$ 必有一个成立

与 $y_1^n = y_2^n = x$ 矛盾，因此 $y^n = x$ 至多有一个 R^+ 解

S_2: 解的存在性

要在R内证 $\exists \beta$, s.t. $\beta = \alpha$，可以利用 R的lub/glb性，构造 $A = \{x \in R/R^+ \mid x < \alpha\}$，然后证

A 非空有上界，则一般会有 $\sup A = \beta$ 成立，即 $\beta = \sup A = \alpha$

令 $A = \{t \in R^+ \mid t^n < x\}$

S_1: $A \neq \phi$

1° $x > 1$，令 $t = 1$ 即可

2° $x \leq 1$，令 $t = \frac{x}{2}$ 即可

S_2: A有上界

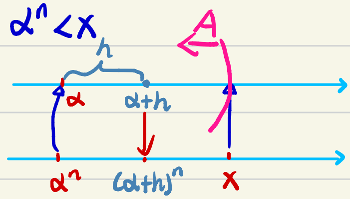
$(x+1)^n > x+1 > x > t^n$

$\therefore \forall t \in A$, $t < x+1$

A存在sup，设 $\sup A = \alpha$

下证 对 $A=\{t\in\mathbb{R}^+ \mid t^n<x\}$, $\alpha=\sup A$, 有 $\alpha^n=x$

1° 若 $\alpha^n<x$



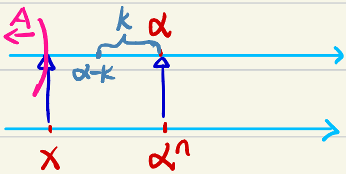
找到增量 h, s.t. $(\alpha+h)^n<x$, 即 $\alpha+h\in A$

而 $\alpha<\alpha+h$, 因此 α 不能为 A 的 u.b. 矛盾

$$h<\frac{x-y^n}{n(y+1)^{n-1}} \Leftarrow b^n-a^n<(b-a)nb^{n-1}$$

$b=y+h$
$a=y$

2° 若 $\alpha^n>x$



找到增量 k, s.t. $(\alpha-k)^n\geq x$ 则 α 非 A 的 sup, 矛盾

$$k=\frac{y^n-x}{ny^{n-1}}$$

故由 $\mathbb{R}$ 的三歧性, $\alpha^n=x$, 即 $y^n=x$ 有解 $y=\alpha$

QED.

Corollary *If a and b are positive real numbers and n is a positive integer, then*

$$(ab)^{1/n}=a^{1/n}b^{1/n}.$$

Proof Put $\alpha=a^{1/n}$, $\beta=b^{1/n}$. Then

$$ab=\alpha^n\beta^n=(\alpha\beta)^n,$$

since multiplication is commutative. [Axiom (M2) in Definition 1.12.] The uniqueness assertion of Theorem 1.21 shows therefore that

$$(ab)^{1/n}=\alpha\beta=a^{1/n}b^{1/n}.$$

四、拓展实数系（不再是field）

1.23 Definition The extended real number system consists of the real field R and two symbols, $+\infty$ and $-\infty$. We preserve the original order in R, and define

$$-\infty < x < +\infty$$

for every $x \in R$.

∀一个 R 中的非空集合，均有 u.b. $+\infty$，故均有 sup（lub性），ub同理

The extended real number system does not form a field, but it is customary to make the following conventions:

(*a*) If x is real then

$$x + \infty = +\infty, \qquad x - \infty = -\infty, \qquad \frac{x}{+\infty} = \frac{x}{-\infty} = 0.$$

(*b*) If $x > 0$ then $x \cdot (+\infty) = +\infty$, $x \cdot (-\infty) = -\infty$.
(*c*) If $x < 0$ then $x \cdot (+\infty) = -\infty$, $x \cdot (-\infty) = +\infty$.

五、复数域

复数的本质实为R上的一个有序数对

复数域上的 Cauthy – Schwarz 不等式（方和积 ≥积和方）

1.35 Theorem *If $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ are complex numbers, then*

$$\left|\sum_{j=1}^{n} a_j \bar{b}_j\right|^2 \le \sum_{j=1}^{n} |a_j|^2 \sum_{j=1}^{n} |b_j|^2.$$